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GAS ION MOBILITIES IN ETHER-HYDROGEN MIXTURES

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Statement of Problem.—The recent quantitative studies of ionic mobilities in mixtures of air and gases of high dielectric constant or high chemical activity by Loeb and Ashley,[1] Loeb,[2] Tyndall and Phillips,[3] and Mayer[4] have yielded very valuable information regarding ionic constitution. They showed that the mobility of the negative or of both ions was abnormally decreased when small quantities of the active gas were added to the air. Except for the measurements by Loeb and Ashley[1] and by Loeb,[2] the observations were not extended over the whole range of concentrations. This was because the low vapor pressures of the vapors used by the other workers or because of the reactive nature of their gases, did not enable them to extend their measurements beyond a few per cent of the active gas.

In the experiments of Loeb, the measurements were carried over the whole range of concentrations so that they could be compared with the theoretical law for mobilities in mixtures. The reactive nature of the gases used made accurate comparison between theory and observation unsatisfactory due to changes of the electrode surfaces. The variations introduced by such accidental effects at the higher concentrations were of the same magnitude as possible real deviations from the theoretical curves; for the variation of mobility with constitution was slight in this region for gases where the mobilities in the pure gas and air did not differ widely to begin with.

It was, therefore, proposed to study mobilities in mixtures of ether vapor and hydrogen. Ether has one of the lowest mobilities known while hydrogen has the highest mobility of any known gas. Furthermore, ether is not reactive and has a high vapor pressure at room temperatures.

Experimental Method.—The method used was the well-known Franck modification of the Rutherford alternating current method. The ionization chamber used was the one used by the writer in his study of the ionic mobilities in ether as a function of the pressure and is there described.[5]

The auxiliary field used was 7.5 volts per cm. and for this reason the absolute mobilities were slightly higher than published in that paper.[6] For ether, the mobilities were found to be 0.21 and 0.23 as against the values 0.19 and 0.22 for positive and negative ions, respectively, found in the earlier work with a lower auxiliary field. The frequency was so adjusted that the value of the alternating field lay in the neighborhood of 25–40 volts per cm. for negative ions and 30 to 50 volts per cm. for positive ions. The relative results should, therefore, be little influenced by the interpenetration of the fields through the gauze.[6]

The ether used was Baker's, U. S. P. ethyl ether, for anesthesia. The method of preparing a mixture was first to exhaust the ionization chamber to X-ray vacuum or better, and then distill in the ether from a sealed bulb,

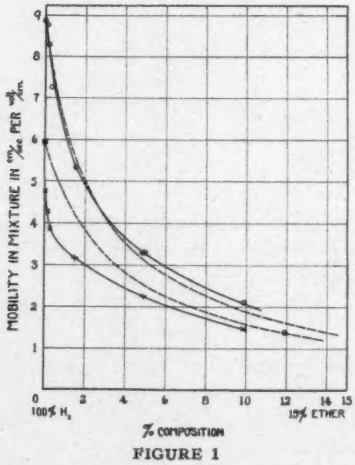

FIGURE 1

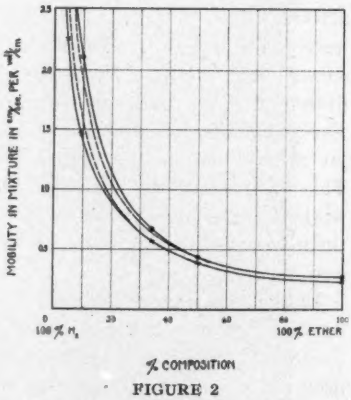

FIGURE 2

reading the pressure on a manometer. H_2 gas from a tank was then run in after passing through a system of KOH, $CaCl_2$ and P_2O_5 tubes and two liquid air traps. The percentages of ether at less than 2 mm. pressure are necessarily only rough. However, the error is not serious and the results are consistent.

All measurements were made at about atmospheric pressure (600 to 760 mm.), at a room temperature of 21° to 23°C. and reduced to 760 mm. except the ones in pure ether. In the latter case, owing to the fact that the vapor pressure of ether at 20°C. is only 400 mm., pressures of 300 mm. were used and the results reduced to 760 mm. in the usual manner.

Results.—The results are summarized in the two sets of curves in figures 1 and 2. Figure 1 gives the curves for positive (lower curve) and negative (upper curve) ions for percentages of ether from 0–20%. Figure 2 gives

the same curves for percentages from 20 to 100% of ether. The full curves are the experimentally observed ones, the actual observations being indicated by the points. The broken curves are the mobilities to be expected in such a mixture computed by the theoretically correct equation first used by Blanc[7] and found by him to apply to mixtures of CO_2, H_2 and air. The equation reads

$$K_c = \frac{K_H \cdot K_{\text{eth.}}}{\frac{(100-c)K_{\text{eth.}} + cK_H}{100}}$$

where K_H is the mobility observed in pure H_2, and $K_{\text{eth.}}$ that in pure ether.

It is seen at once that for negative ions the two curves lie close together, in fact, the observed points deviate from the theoretical curve by amounts of the order of experimental uncertainty only. For the positive ions this is true only above about 30% of ether. The addition of a small trace of ether lowers the mobility appreciably. This is the same effect that was observed in HCl for positive and negative ions by Loeb,[2] in the alcohol and water vapors by Tyndall and Phillips[3] and in Cl_2 for negative ions by Mayer.[4] The effect is quite striking and emphasizes once again the selective change in concentration of one of the gases about the ions, or its selective incorporation into the ion cluster if a stable cluster forms. It also explains why sometimes in making control observations on air the mobility of the positive ions was found by the writer to be abnormally low, while the negative mobility was normal. Ether was usually used in cleaning the stopcocks and was thus sometimes in the apparatus in appreciable amount.

At the higher concentrations of ether both positive and negative ions follow the theoretical law of mixtures. It was this point which appeared likely in HCl-air mixtures but whose certainty was impaired by the experimental difficulties mentioned in the introduction.

Discussion and Conclusions.—In conjunction with the discussion of the question of the theoretical equations for ionic mobilities by Loeb,[8] and with the experiments of Erikson[9,10] on the aging of ions, the above experiments could be interpreted as leading to the following conclusions. In order of magnitude the mobilities of ions in pure gases can be computed from the kinetic theory assuming an inverse fifth power law of force of the form

$$F = \frac{(D-1)e^2}{2\pi N r^5},$$

where D is the dielectric constant, e is the charge on the ion, N is the number of molecules in a cm.^{3}, and r is the distance between ion and

neutral molecule. In such computations, it makes comparatively little difference whether the mean free path be shortened by cluster formation, or by the deviation of the molecular and ionic free paths from straight lines due to the forces acting. Probably the best equation expressing this part of the action can be found in the Langevin[11] equation, or its later modification by Hassé.[12]

Actually, however, the inverse fifth power law breaks down when the ion and molecule approach to within distances of the order of a molecular diameter of each other. In this case, the distances between the separate electrical constituents in the molecules and the ion are of the same order as the distances between the ion and molecule and the law should fail. The recent work has shown that the effect of this failure is to cause changes in mobility of the order of those between positive and negative ions, which, from the beginning, could not very well be explained on the law of force assumed. A plausible qualitative explanation proposed by Condon,[13] showing how differences in molecular constitution affect the positive and negative ion mobilities is published elsewhere.

Under the ordinary conditions of study of mobilities in pure gases, the types of electrical activity causing aging or the selective concentration of molecules about an ion are not obvious. When experiments are conducted under conditions where the ion has had no time to alter the conditions in its neighborhood (either to increase the concentration of the active molecules near it, or to actually attach such molecules), one obtains the changes in mobility observed by Erikson. Where the active molecules are present in small relative numbers, the same effect manifests itself in an abnormal change of mobility such as observed in the above experiments. The results of Erikson on the aging of positive ions are thus in accord with the results obtained. This is confirmed by the fact that the writer in a previous study,[5] had found an aging effect for positive ions in ether.

The results are further interesting in that they show a case where the negative ion is unaffected in a gas of high dielectric constant, and follows the ordinary law of mixtures while the positive ion shows the abnormal effect at low concentrations, at the same time following the usual law in mixtures so rich in ether that the ion does not change the composition of the gas in its neighborhood appreciably. To what extent such results may be considered to prove the existence of a stable cluster is doubtful. They do definitely show changes in concentration of acting molecules and, in Tyndall and Phillips' results,[3] they show the effect of the size of such molecules on the mobility. This is a great step in advance. The writer's reason for hesitating at a decision concerning the existence of a stable cluster results from a recent analysis of the question of the recombination of ions[14] which show a considerable reluctance at N. T. P. for ions to re-

combine even when the forces acting are due to forces of the type e^2/r^2, instead of

$$\frac{(D-1)e^2}{2\pi Nr^5}$$

which are much weaker.

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[2] Loeb, L. B., *Ibid.*, **12**, 35 and 42 (1926).
[3] Tyndall and Phillips, *Proc. Roy. Soc.*, **A111**, 577 (1926).
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[7] Blanc, A., *J. Physique*, **7**, 825 (1908).
[8] Loeb, L. B., *J. Franklin Inst.*, **201**, 279 (1926).
[9] Erikson, H. A., *Physic. Rev.*, **20**, 117 (1922); **24**, 502 and 622 (1924).
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[11] Langevin, P., *Ann. Chim. Physique* (8), **5**, 238 (1905).
[12] Hassé, H. R., *Phil. Mag.*, **1**, 139 (1926).
[13] Loeb, L. B., *J. Franklin Inst.*, **201**, 285 (1926).
[14] Loeb, L. B., *Kinetic Theory of Gases*, McGraw-Hill, New York, 1927, Chapter XI, "Section on Recombination of Ions."

A REFINEMENT OF THE MICHELSON-MORLEY EXPERIMENT

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The investigation which is the subject of this paper was undertaken to test the conclusions recently drawn by Professor D. C. Miller from his repetition of the Michelson-Morley experiment. Professor Miller interprets his observations as indicating a motion of the sun through the ether with a velocity not less than 200 kilometers per second in a direction about right ascension 262 degrees, declination 65 degrees. He supposes that a contraction of the apparatus in the direction of motion occurs, which, however, departs from that given by the formula of Lorentz and Fitzgerald by an amount corresponding to a velocity of about 10 kilometers per second. That is, the average shift of the interference pattern, as an arm of his interferometer is turned through 90 degrees from the direction perpendicular or parallel to the direction of the inferred drift, is such as would be expected if the velocity of the apparatus were 10 kilometers per second and no contraction occurred. In his paper in *Science*, April 30, 1926, he seems to have abandoned the idea that the magnitude of the

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indicated velocity depends on the altitude above sea level at which the observations are made. Such results as his, modifying as they would the fundamental physical concepts, require the fullest experimental confirmation.

On the classical ether hypothesis the well-known theory of the experiment predicts a shift of the interference pattern proportional to the lengths of the paths traversed by the interfering beams. In order to make a small velocity observable, Professor Miller used a large interferometer in which the light path was about 65 meters long. The difficulty of maintaining sufficient freedom from air currents and temperature effects with so large an instrument can be appreciated when one considers that the shift corresponding to 10 kilometers per second could also be brought about by a change in optical length of one path of less than one part in 10^9. As slight a difference of average density of air along the two arms as would be produced by a pressure difference of 2×10^{-3} mm. of mercury or a temperature difference of 10^{-3} degrees C. would suffice to yield effects of the magnitude observed.

In the present work the light paths were reduced to about 4 meters, and the required sensitiveness obtained by an arrangement capable of detecting a very slight displacement of the interference pattern. The whole optical system was enclosed in a sealed metal case containing helium at atmospheric pressure. Because of its small size the apparatus could be effectively insulated, and circulation and variations in density of the gas in the light paths nearly eliminated. Furthermore, since the value of $\mu-1$ for helium is only about one-tenth that for air at the same pressure it will be seen that the disturbing effects of changes in density of the gas correspond to those in air at only a tenth of an atmosphere of pressure. Actually it was found that any wavering of the interference pattern was imperceptible, and when temperature equilibrium had been reached there was no steady shift.

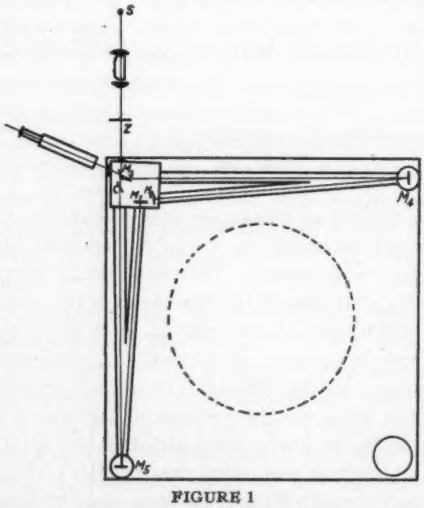

FIGURE 1

The plan of the apparatus is sketched in figure 1. The optical parts are mounted on a marble slab 122 cm. square by 10.5 cm. thick, which

rests on an annular float in a pan of mercury 77 cm. in diameter. This is simply a reduced copy of Michelson's original mounting. The mirrors M_1, M_4 and M_5 are fixed in position; such adjustments of the compensating plate C and mirror M_2 as are necessary after the cover is in place can be made from the observer's position at the telescope. The green light λ 5461 is separated by the lens and prism system from the radiation of a small mercury arc lamp S attached to the slab, and passed through a small hole in the screen Z. The pencils of light are carefully limited by screens and by focusing in order to prevent stray light from reaching the eye and thereby reducing its sensitiveness. Adjustments are made so that broad fringes are formed at the surface of M_1 and M_2, on which the telescope is focused. Final adjustments are made by rotating the compensating plate C by means of a fine differential screw, and by placing small weights near the corner of the slab; under proper conditions a 5 g. weight deflects the heavy slab just perceptibly. The adjusting screws are manipulated by means of spindles passing through short flexible tubes in such a way as to be freely rotatable but air-tight. After the mirrors are given preliminary alignment, the cover is carefully lowered into place, sealed to the slab, and then filled with helium.

Schematically the arrangement of the interferometer is shown in figure 2. A beam of practically plane-parallel, homogeneous light, plane polarized so that its electric vector lies in the plane of the paper, moves to the right and falls on the mirror M_3 at the polarizing angle for the given wavelength. At the upper face the beam is split by a thin platinum film into two parts of nearly equal intensity, one passing to the mirror M_1 and the other to M_2. From there they are reflected back to M_3 where they recombine and pass to the eye through a telescope focused on M_1 and M_2. Two purposes are accomplished by the use of plane polarized light; first, the non-interfering rays indicated by the dotted lines, which would be produced with natural light, are completely eliminated, and second, the recombining beams can be adjusted to perfect equality of intensity by varying the relative reflecting powers of M_1 and M_2. Because there are two more glass-air interfaces to be traversed by the upper beam than the lower, it is impossible to equalize both components of natural light in this way.

The high sensibility necessary because of the short paths is secured chiefly by the simple device of raising one half of the surface of mirror M a small fraction of a wave-length above the other, the dividing line between the two levels being straight and as sharp as possible. The mirror used was made by covering part of a plane plate with a flat sharp-edged microscope cover-glass and applying the extra thickness by cathode deposition of platinum; thereafter giving the whole plate a fully-reflecting coat. The writer ran across the suggestion of using such a divided mirror in in-

terferometry some years ago, but is unaware to whom the credit for it belongs.

The theory of the arrangement is as follows: The interference phenomena will be the same as if the mirror M_2 were replaced by its image in M_3. Under the conditions of the experiment, where the paths are nearly equal, M_1 is perpendicular to the beam incident on it, and the reflected beams are brought nearly to parallelism, the image of M_2 will be nearly parallel and coincident with the face of M_1. Elementary theory shows then that the resulting interference pattern will practically coincide with M_1. It would needlessly complicate this discussion to develop the general theory of interference for all inclinations of the mirrors; the experimentally realized case of near parallelism alone is necessary.

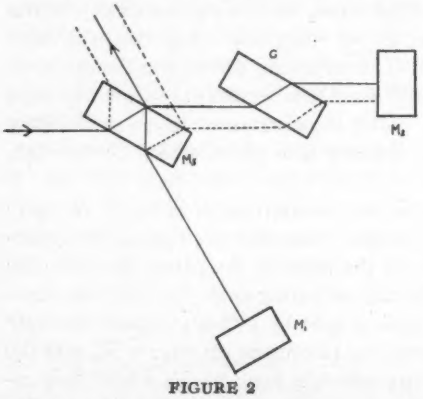

FIGURE 2

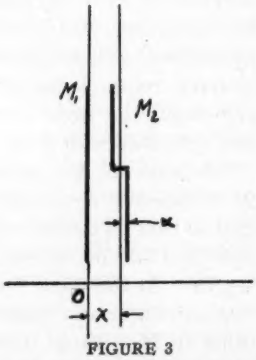

FIGURE 3

Let figure 3 represent a greatly exaggerated cross-section of M_1 and the image of M_2, normal to their planes and to the dividing line in $M_2 \ldots M_1$ lies in the plane $x = 0$, and the levels of M_2 are at equal distances on opposite sides of a parallel plane at the distance x from M_1. Let a monochromatic wave, in which the displacement is given by

$$\xi = a \cos \omega \left((t + \epsilon - \frac{x}{c} \right),$$

fall on M_1 and M_2 from the left. At the surface of M_1 the displacement in the reflected wave is then given by

$$\xi_1 = a \cos \omega \, (t + \epsilon)$$

if we ignore the loss through imperfect reflection. The displacement in the plane of M_1 in the wave reflected from the upper part of M_2 is

$$\xi_2 = a \cos \omega \left[t + \epsilon - \frac{2(x - \alpha)}{c} \right].$$

The square of the resultant displacement is then

$$(\xi_1 + \xi_2)^2 = a^2 \left\{ \cos \omega (t + \epsilon) + \cos \omega \left[t + \epsilon - \frac{2(x - \alpha)}{c} \right] \right\}.$$

This can be reduced to the form

$$2a^2 \left[1 + \cos \frac{2\omega}{c} (x - \alpha) \right] \cos^2 \omega (t - \delta).$$

Similarly the square of the resultant displacement in the interfering beams below the dividing line is found to be

$$2a^2 \left[1 + \cos \frac{2\omega}{c} (x + \alpha) \right] \cos^2 \omega (t - \delta).$$

The intensities, being proportional to the squares of the amplitudes, can be represented by

$$I_1 = ka^2 \left[1 + \cos \frac{2\omega}{c} (x - \alpha) \right]$$

and

$$I_2 = ka^2 \left[1 + \cos \frac{2\omega}{c} (x + \alpha) \right].$$

Now $\omega = 2\pi\nu$ where ν = frequency of the light. Hence $\frac{\omega}{c} = \frac{2\pi}{\lambda}$.

$$\therefore I_1 = ka^2 \left[1 + \cos \frac{4\pi}{\lambda} (x - \alpha) \right]$$

and

$$I_2 = ka^2 \left[1 + \cos \frac{4\pi}{\lambda} (x + \alpha) \right].$$

For values of $x = \frac{n\lambda}{4}$, where n is an integer,

$$I_1 = ka^2 \left(1 \pm \cos \frac{4\pi\alpha}{\lambda} \right)$$

the sign being positive for even values of n, and negative for odd values. The same expression holds for I_2; hence under these conditions

$$I_1 = I_2.$$

To the observer, then, the field of view is equally intense on both sides of the dividing line when $x = \frac{n\lambda}{4}$.

We have now to determine the least change in x from this value which will produce a perceptible difference in illumination in the two sides of the

field. If x is given the variation δx while α is kept constant, the difference in intensity will be

$$\delta I = \left(\frac{\partial I_1}{\partial x} - \frac{\partial I_2}{\partial x}\right)\delta x.$$

Now
$$\frac{\partial I_1'}{\partial x} = -\frac{4\pi ka^2}{\lambda}\sin\frac{4\pi}{\lambda}(x-\alpha)$$

$$= \pm\frac{4\pi ka^2}{\lambda}\sin\frac{4\pi\alpha}{\lambda}.$$

Similarly

$$\frac{\partial I_2}{\partial x} = \pm\frac{4\pi ka^2}{\lambda}\sin\frac{4\pi\alpha}{\lambda}.$$

$$\therefore \delta I = \pm\left[\frac{8\pi ka^2}{\lambda}\sin\frac{4\pi\alpha}{\lambda}\right]\delta x,$$

the sign being of no importance.

The perceptibility of the variation is determined not by δI alone, but by the ratio of δI to the total intensity, I_1 or I_2. According to the Weber-Fechner law, if δI is taken to be the least perceptible variation in intensity, the above ratio is nearly constant for a considerable range of intensities. With this meaning of δI, δx becomes the least detectable change of position of M_2.

If we have initially uniformity of illumination, we have from the equations above,

$$\frac{\delta I}{I} = \frac{8\pi}{\lambda}\delta x\,\frac{\sin\frac{4\pi\alpha}{\lambda}}{1\pm\cos\frac{4\pi\alpha}{\lambda}},$$

or

$$\delta x = \frac{\lambda}{8\pi}\frac{\delta I}{I}\,\frac{1\pm\cos\frac{4\pi\alpha}{\lambda}}{\sin\frac{4\pi\alpha}{\lambda}}.$$

If now $\frac{\delta I}{I}$ were a true constant we should have for the case of negative sign, which corresponds to dim illumination of the field, the sensibility of the apparatus increasing indefinitely as the factor α was made smaller. Unfortunately, however, I decreases with α and the Fechner "constant" soon diminishes rapidly. Nevertheless, the conditions of illumination

and contrast here are similar to those in the half-shade polariscope, and from the theory of the Lippich instrument it appears that $\frac{\delta I}{I}$ equals about 8×10^{-3}. The lack of perfect planeness in the mirrors and of equality of intensity in the interfering beams is a further limiting factor; a little experimenting indicated that α should be not much less than 0.025λ, which was the value finally used. Substituting these values in the last equation we get

$$\delta x = 5 \times 10^{-5}\lambda$$

as the least detectable change in position of one of the mirrors. This corresponds to a change of optical length of path

$$\delta l = 2\delta x = 10^{-4}\lambda.$$

To take full advantage of the possibilities of the arrangement would have required perfect mirrors and an intenser and, therefore, hotter source of light than would have been desirable near the sensitive apparatus, as well as lengthening the interval between observations and, thereby allowing greater chance for any steady temperature shift to show itself. It was, therefore, not attempted in the experiment to go below values of δl equal to $2 \times 10^{-3}\lambda$; such variations were detectable without the least uncertainty.

The theory of the Michelson-Morley experiment is given too often to need discussion here. It turns out that as the apparatus is rotated through a right angle the interference pattern should be shifted as if by a change of length of one of the paths by a fraction of a wave-length whose maximum is

$$\delta l = \frac{l}{\lambda}\frac{v^2}{c^2}\cos^2\beta. \qquad (1)$$

Here v is the velocity of the apparatus with respect to the ether, l is the length of light path and β is angle between the plane of the apparatus and the direction of motion.

Suppose that in a set of rectangular coördinates, x_1, x_2, x_3 fixed in the ether the direction cosines of the velocity vector of the supposed ether drift are α_1, α_2, α_3, while the direction cosines of the normal to the plane of the apparatus are α_1', α_2', α_3'. Then the angle between the two lines will be given by

$$\cos\theta = \Sigma\,\alpha_i\,\alpha_i' \qquad (i = 1, 2, 3). \qquad (2)$$

Also $$\alpha_j = \frac{x_j}{(\Sigma\, x_i^2)^{1/2}} \quad \text{and} \quad \alpha_j' = \frac{x_j'}{(\Sigma\, x_i'^2)^{1/2}}. \qquad (3)$$

But with a suitable orientation of the Cartesian system, we have

$$x_1 = r \cos \psi \cos \varphi$$
$$x_2 = r \sin \psi$$
$$x_3 = -r \sin \varphi \cos \psi$$

where φ is the right ascension and ψ the declination of the point where the given direction cuts the celestial sphere. A similar set of equations relates the x''s with angles φ' and ψ'. Substituting these relations in equations (3) and the resulting relations in equation (2) we get

$$\cos \theta = \cos \psi \cos \psi' \cos (\varphi - \varphi') + \sin \psi \sin \psi'. \qquad (4)$$

The angle β in formula (1) is evidently the complement of θ so that δl is a maximum when $\cos \theta$ is a minimum. From (4) it is seen that this occurs when

$$\varphi' = \varphi + n\pi,$$

n being an odd integer. Substituting this value and that of $\psi' = 34° 8'$, the latitude of Pasadena, together with the values $\varphi = 262°$ and $\psi = 65°$ determined by Miller, in equation (4) the minimum of $\cos \theta$ is found to be about 0.15; consequently, the maximum of $\cos \beta$ is very nearly unity. This occurs at a sidereal time equal to $\frac{1}{15} \varphi' = \frac{262-180}{15} = 5.5$ hrs., which during the middle two weeks of September, when the present work was done corresponds to mean local solar times varying from 6.30 A.M. to 5.30 A.M.

If in equation (1) the values $l = 400$ cm., $v = 10^6$ cm. per sec., $c = 3 \times 10^{10}$ cm. per sec., $\lambda = 5.46 \times 10^{-5}$ cm., and $\cos \beta = 1$ are substituted, we find for the maximum shift to be expected with this apparatus an amount corresponding to a change in one of the paths

$$\delta l = 8 \times 10^{-3} \text{ wave-lengths},$$

which is four times the least amount detectable.

The experiment was performed in a constant-temperature room in the Norman Bridge Laboratory at various times of day, but oftenest at the time when Miller's conclusions require the greatest effect. The sensitiveness of the eye was tested for each trial by the placing or removal of a small weight on the slab before and after rotating it. There being no fluctuations in the field of view, it was unnecessary to take the average of a number of readings. As has been shown, a shift as small as one-fourth that corresponding to Miller's would be perceived. The result was perfectly definite. There was no sign of a shift depending on the orientation.

Because an ether drift might conceivably depend on altitude, the experiment was repeated at the Mount Wilson Observatory, in the 100-inch telescope building. Here again the effect was null.

It is intended to make a systematic search for a possible drift in other directions when the apparatus has been slightly modified to increase its sensitiveness and facilitate the numerous observations that will be necessary.

The writer is under great obligation to Dr. R. A. Millikan, whose continued interest has made this investigation possible.

ON THE EVALUATION OF CERTAIN INTEGRALS IMPORTANT IN THE THEORY OF QUANTA

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Communicated September 22, 1926

1. It is known[1] that the matrix of the hydrogen atom, determining the intensities of the hydrogen series lines and of their fine structure components, essentially depends on the integral

$$I = \int_0^\infty r^3 \chi \chi' dr, \tag{1}$$

where the functions χ and χ' are solutions of the Schroedinger equation[2]

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left(\frac{2\mu E}{K^2} + \frac{2\mu e^2}{K^2 r} - \frac{(k-1)k}{r^2}\right)\chi = 0. \tag{2}$$

The symbols μ, e stand for the mass and the charge of the electron; $K = h/2\pi$ (h Planck's constant). The energy E and the integer k have different values in χ and χ'. These functions are supposed to be finite for $r = 0$ and to vanish for $r = \infty$.

Since in the case of elliptic orbits the functions χ, χ' turn out to be polynomials multiplied by an exponential, the direct evaluation term by term is possible on principle. The numerical computations involved are, however, so lengthy as to make this method almost prohibitive in practice. We give, therefore, in this note a reduction of the integral (1) to a simple and convenient expression. Such a reduction is quite indispensable in the case of hyperbolic orbits. Moreover, the procedure applied has an interest beyond the special case of the Kepler motions, since quite analogous expressions occur in other problems of the quantum theory. In fact, the same method has been used by the author for reducing the intensity expressions of the components in the Stark effect.[3] Only the simple closed Kepler motion, neglecting relativity effect, and spin of the electron, will be considered in the following sections.

2. Our procedure is based on the existence of certain recurrent relations between functions χ with different parameters. We use the substitution

$$\chi = r^{k-1} \exp.(\alpha r).M, \qquad \alpha^2 = -2\mu E/K^2. \tag{3}$$

According to Schroedinger, only values of α are permissible for which

$$\mu e^2/\alpha K^2 = -(s + k) = -l, \tag{4}$$

where s and l are two other integers. Equation (2) is transformed into

$$\frac{d^2M}{dr^2} + 2\left(\alpha + \frac{k}{r}\right)\frac{dM}{dr} - \frac{2\alpha s}{r} M = 0. \tag{5}$$

The solution of this equation, satisfying our requirements of finiteness obviously is

$$M = 1 + \frac{s}{1.2k} 2\alpha r + \frac{s(s-1)}{1.2.2k(2k+1)} (2\alpha r)^2 + \ldots. \tag{6}$$

We can consider this expression as derived from a hypergeometric function by a process of transition to the limit

$$M(l, 2k) = \lim_{\substack{x=0,\ \beta=\infty \\ \beta x = r}} F(-s, \beta, 2k, -2\alpha x). \tag{7}$$

I.e., we let x decrease, β increase indefinitely, keeping, however, βx finite and equal to r. The advantage of this representation is that from the relations existing between hypergeometric functions[4] we can immediately derive the following recurrences for M:

$$(2k + s - 1)M(s, 2k) - sM(s - 1, 2k) = (2k - 1)M(s, 2k - 1), \tag{8}$$

$$2\alpha r M(s, 2k) = (2k - 1)[M(s + 1, 2k - 1) - M(s, 2k - 1)], \tag{9}$$

$$2\alpha r M(s, 2k) = (2k + s)M(s + 1, 2k) - (2k + 2s)M(s, 2k) + sM(s-1, 2k), \tag{10}$$

$$\frac{dM(s, 2k)}{dr} = \frac{s}{r}[M(s, 2k) - M(s - 1, 2k)]. \tag{11}$$

The parameter α is supposed to have the same value in all the functions M of these formulae.

3. We shall consider the case in which the function χ depends on the arguments s and α, χ' on s', α' while k is the same in both. Therefore, we shall omit in this section the parameter k in order to abbreviate our symbols. Moreover, for the present, we shall not make use of the connection (4) between α and s, k. If we denote for short

$$R(s, s') = \int_0^\infty r\chi(s, \alpha)\chi(s', \alpha')dr, \tag{12}$$

we can obtain a relation from equation (2). Multiplying that equation by $r^2\chi'$, subtracting the corresponding equation for χ' multiplied by $r^2\chi$, and integrating with respect to r from 0 to ∞, we have

$$(\alpha^2-\alpha'^2)\int_0^\infty r^2\chi(s,\alpha)\chi(s',\alpha')dr + 2[(s+k)\alpha-(s'+k)\alpha']R(s,s') = 0. \tag{13}$$

If we transform the integral by means of our recurrence (10) and use the substitutions

$$R(s,s') = u^s V(s,s')/(\alpha+\alpha')^{2k}, \qquad u = (\alpha-\alpha')/(\alpha+\alpha') = (l-l')/(l+l'), \tag{14}$$

this relation can be written so

$$(s+2k)u^2V(s+1) - [(s-s') + (s+s'+2k)u^2]V(s) + sV(s-1) = 0. \tag{15}$$

We do not write the argument s' because in this relation (as in the following ones) it is kept constant. The interest of relation (15) lies in the fact that by means of it we can compute successively all the $V(s, s')$ if only $V(0, 0)$ is known. According to (3) and (6)

$$R(0,0) = \int_0^\infty r^{2k-1}\exp.[(\alpha+\alpha')r]dr = \frac{(2k-1)!}{(\alpha+\alpha')^{2k}}; \qquad V(0,0) = (2k-1)! \tag{16}$$

It follows that $V(s)$ depends on $u^2 = z$ only. Noticing this and bearing in mind that M is a function of the product αr, we get a second relation by differentiating (12) with respect to α:

$$2z(1-z)\frac{dV(s)}{dz} = -(s+2k)zV(s+1) - [s-(s+2k)z]V(s) + sV(s-1). \tag{17}$$

Combining (15) and (17) we obtain a differential equation determining $V(s, s')$

$$4z^2(1-z)\frac{d^2V(s)}{dz^2} + 4z[(s+1)-(s+2k+1)z]\frac{dV(s)}{dz} + s[(2s-s') + (s'+2s+4k)z]V(s) = 0. \tag{18}$$

4. This last differential equation is of the hypergeometric type and its solution obviously is

$$V(s,s') = Cu^{-s'}F(s+2k, -s', s-s'+1, u^2) \tag{19}$$

where C is a constant depending on s, s', k. Its dependence on s can be obtained from (15) and is given by the factor $s!/(s-s')!$. The rest can be found from the limiting value (16) and from the requirement of symmetry

with respect to s and s'. We obtain in this way $C = (2k-1)!\,(2k-1)!\cdot s!/(s-s')!(s'+2k-1)!$ giving

$$R(s, s') = \frac{(2k-1)!\,(2k-1)!\,s!}{(s-s')!\,(s'+2k-1)!} \cdot \frac{u^{s-s'}}{(\alpha+\alpha')^{2k}} \cdot F(s+2k, -s', s-s'+1, u^2). \quad (20)$$

The symmetry of this expression is brought out if we rearrange it in falling powers of u^2

$$R(s, s') = (-1)^{s'} \frac{(2k-1)!\,(2k-1)!\,(s+s'+2k-1)!}{(s+2k-1)!\,(s'+2k-1)!} \cdot \frac{u^{s+s'}}{(\alpha+\alpha')^{2k}} \cdot F\left(-s, -s', -s-s'-2k+1, \frac{1}{u^2}\right). \quad (21)$$

For finding the intensities we must normalize the functions χ. To do this we have to know the value of the integral $\int_0^\infty r^2\chi^2 dr$. We can obtain it by applying (10) to $r\chi$. Since in our case $\alpha = \alpha'$, $u = 0$, the hypergeometric function is reduced to unity. Moreover $u^{s-s'}$ is equal to 1 in the middle term and vanishes in the two other terms, so that

$$1/A(k, s) = \int_0^\infty r^2\chi^2(s, k)dr = (-2\alpha)^{-2k-1}\, 2(s+k)\,(2k-1)! \cdot (2k-1)!\,s!/(s+2k-1)! \quad (22)$$

5. The intensity expressions contain as their essential factor the integral

$$I = \sqrt{A(k, s)A(k-1, s')} \int_0^\infty r^2\chi(s, k)\chi(s', k-1)\, dr. \quad (23)$$

We apply (9) two times

$$I = \sqrt{A(k, s)A(k-1, s')}\, \frac{(2k-2)(2k-1)}{4\alpha^2} \cdot \int_0^\infty r^2[\chi(s, k-1) - -2\chi(s+1, k-1) + \chi(s+2, k-1)]\chi(s', k-1)dr. \quad (24)$$

Relation (13) is always valid, except when $\alpha = \alpha'$, $s = s'$. Since, in our case, $1/\alpha \backsim s+k$, $1/\alpha' \backsim s'+k-1$, we may apply (13) to all three terms of expression (24). Using abbreviations (14) and relation (15), we arrive at

$$I = -\frac{(1-u^2)^{k-1/2}\, u^{s-1}}{2(\alpha+\alpha')(2k-3)!(2k-3)!} \sqrt{\frac{(s+2k-1)!\,(s'+2k-3)}{(s+k)(s'+k-1)s!\,s'!}} \cdot [V(s, s') - u^2 V(s+2, s')]. \quad (25)$$

If we substitute expressions (21)

$$I(l,k;l',k-1) = (-1)^{s'}\frac{(1-u^2)^{k-1/2}\,u^{s+s'-1}}{2(\alpha+\alpha')\sqrt{l.l'}}\cdot$$

$$\cdot\frac{(l+l'-2)!}{\sqrt{s!\,s'!(l+k-1)!\,(l'+k-2)!}}\cdot\{u^2(l+l'-1)(l+l')\psi(s+2',s',k)-$$

$$-(l'+k-2)(l'+k-1)\psi(s,s',k)\} \quad (26)$$

using the abbreviation

$$\psi(s,s',k) = F\left(-s,-s',-s-s'-2k+3,\frac{1}{u^2}\right). \quad (27)$$

By an analogous reasoning we obtain for the case $k' = k+1$ the expression

$$I(l,k;l',k+1) = (-1)^{s'+1}\frac{(1-u^2)^{k+1/2}\,u^{s+s'-1}}{2(\alpha+\alpha')\sqrt{l.l'}}\cdot$$

$$\cdot\frac{(l+l'-2)!}{\sqrt{s!\,s'!(l+k-1)!\,(l'+k)!}}\cdot\{u^2(l+l'-1)(l+l')\psi(s,s'+2,k+1)-$$

$$-(l'+k-1)(l'+k)\psi(s,s',k+1)\}\cdot \quad (28)$$

6. In the case of the Balmer series we have $l' = 2$ with the two possible cases $k' = 2, s' = 0$ and $k' = 1, s' = 1$. We arrive at the following three possible values

$$\left.\begin{aligned} I(l,3;2,2) &= -B\cdot\frac{2l}{l^2-4}\sqrt{\frac{(l+2)!}{3!(l-3)!}},\\ I(l,2;2,1) &= \quad B\cdot 2\sqrt{\frac{(l+1)!}{2!(l-2)!}},\\ I(l,1;2,2) &= \quad B.l\sqrt{\frac{l!}{3!(l-1)!}}\\ B &= (1-u^2)^{1/2}\,u^{l-3}/(\alpha+\alpha')\sqrt{2l}. \end{aligned}\right\} \quad (29)$$

These formulae give, in fact, numerical values identical with those found by Eckart.

[1] C. Eckart, *Physic. Rev.*, October, 1926; E. Schroedinger, *Ann. Physik*, **80**, p. 437, 1926.

[2] E. Schroedinger, *Ann. Physik.*, **79**, p. 361, 1926.

[3] P. S. Epstein, *Physic. Rev.*, October, 1926.

[4] Compare: W. Laska, *Sammlung von Formeln*, p. 45, Braunschweig, 1888–1894.

THE NEW QUANTUM THEORY AND THE ZEEMAN EFFECT

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1. The treatment of the Zeeman effect from the point of view of Schroedinger's theory presents certain difficulties. Let us consider the case of a hydrogen atom in a homogeneous magnetic field of the strength **H**. We start from the customary expressions of the Hamiltonian function for this case, and derive from it the fundamental wave equation by the procedure given by Schroedinger[1] and independently by Eckart.[2] The result is the equation given explicitly by Eckart

$$\nabla^2\psi + \frac{e}{ick}\,\mathbf{H}\cdot(\mathbf{r}\times\nabla\psi) + \frac{2\mu}{k^2}\left(E + \frac{e^2}{r}\right)\psi = 0, \qquad (1)$$

e and μ denote the charge and mass of an electron, c the velocity of light, E the energy constant, k is an abbreviation for $h/2\pi$ (h Planck's constant) and the vector $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$.

If we introduce a system of polar coördinates r, ϑ, φ, taking the direction of the magnetic field as the polar axis, the equation becomes

$$[\psi] + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{eH}{ick}\frac{\partial\psi}{\partial\varphi} = 0, \qquad (2)$$

with the abbreviation

$$[\psi] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right) + \frac{2\mu}{k^2}\left(E + \frac{e^2}{r}\right)\psi. \qquad (3)$$

The difficulty above mentioned lies in the fact that the term with $\partial\psi/\partial\varphi$ is imaginary. Formally the equation can be solved by substituting $\psi = \psi'$ exp. $(\pm i\tau\varphi)$. This even leads to energy levels identical with those of the old theory, but the factor exp. $(\pm i\tau\varphi)$ does not represent a standing wave required by Schroedinger's theory. If we take into account the time factor exp. $(i\omega t)$ of the partial oscillations ($\omega = 2\pi\nu$), we get the factor exp. $(i\omega t \pm i\tau\varphi)$, which represents a progressing wave winding itself around the polar axis at the very high angular velocity ω/τ.

That this is impossible becomes apparent also from the following consideration. Before subjecting the Hamiltonian function to the Schroedinger procedure, we could have transformed it by introducing rotating axes of coördinates. If the rotation is that of the Larmor precession, the transformed Hamiltonian function is identical with that of a resting hydrogen atom not acted upon by a magnetic field. Since the Hamiltonian function defines Schroedinger's wave equation, we see, carrying out this process,

that in the Zeeman effect the system of standing waves is the same for the rotating observer as in the simple case without a magnetic field for the resting observer. In other words, if we subject the wave system, existing in the simple case, to a Larmor precession, we obtain the solution in the case of the Zeeman effect. Our difficulty can now be stated by the following question: Why cannot we reverse the order of operations? Why is it, that carrying out the Schroedinger process first, we find equation (1) whose solution has nothing to do with that stated above? We shall see in the following sections that equation (1) is incorrect and that the problem must be approached in a different way.

2. It appears that the difficulty stated in section 1 is not a new one, but only an aggravated form of an old trouble of the theory of the Zeeman effect under which it labored from the very beginning of the quantum theory. The magnetic forces are not conservative and the integration constant of the dynamical equations for a system moving in a magnetic field does not represent the whole of the magnetic energy. Early workers in this field were puzzled why, taking into account only this part of the energy, they were led to correct results. It was pointed out by H. A. Lorentz[3] that one can get around the difficulty by extending the system under consideration and by including into it the source of the magnetic field. The extended system will obey the law of conservation of energy and the ambiguity will be removed. We shall show that Lorentz's suggestion solves also the discrepancy in our case.

Let us use as the source of our magnetic field a current J moving, without resistance, in a closed linear conductor of uniform cross-section. We denote by q the position of a chosen particle of charge in this conductor and by $\dot{q}$ the uniform velocity of all the charges in it. If the linear density of charge is ρ and the coefficient of self-induction L, we have $J = \rho\dot{q}$, while the energy of the current is

$$LJ^2/2 = L\rho^2\dot{q}^2/2. \tag{4}$$

Denoting for short by T and U the kinetic and potential energy of the electron in our hydrogen atom, we have for the total kinetic potential of the system, neglecting the interaction between atom and current,

$$K_0 = T + L\rho^2\dot{q}^2/2 - U. \tag{5}$$

To this must be added the contribution of the interaction. The electron moves with the velocity $\mathbf{v}$ in the magnetic field of the current, which can be described by the vector potential $\mathbf{A}$. This gives rise to the term $K' = -e(\mathbf{v}\cdot\mathbf{A})/c$. We could write for the vector potential $\mathbf{A} = \dot{q}\mathbf{A}_1$, where the vector $\mathbf{A}_1$ is independent of $\dot{q}$. Similarly we can write for the strength of field $\mathbf{H} = \dot{q}\mathbf{H}_1$, with the relation $\mathbf{H}_1 = \nabla \times \mathbf{A}_1$. Moreover, making use of the indeterminateness of the vector potential, we subject

it to the conditions: $\mathbf{A}_1 = 0$, in the position of the nucleus, and $\nabla \cdot \mathbf{A}_1 = 0$, generally. Finally, we must remember that we are dealing with the case that the field can be regarded as homogeneous over the size of the atom, so that $\mathbf{H}_1$ is constant and $\mathbf{A}_1 = (\mathbf{H}_1 \times \mathbf{r})/2$. Therefore, we have

$$K' = -e\mathbf{v}\cdot(\mathbf{H}_1 \times \mathbf{r})\dot{q}/2c = e\mathbf{H}_1\cdot(\mathbf{r} \times \mathbf{v})\dot{q}/2c.$$

On the other hand the current is flowing in the magnetic field of the electron. It is, however, well known that this fact does not involve a new term in the kinetic potential. It only leads to a different formulation of K' and is taken care of by this term. The complete kinetic potential is, therefore,

$$K = \mu v^2/2 + L\rho^2\dot{q}^2/2 + e\mathbf{H}_1\cdot(\mathbf{r} \times \mathbf{v})\dot{q}/2c. \qquad (6)$$

3. Deriving from this the momenta and the Hamiltonian function in the usual way, we find

$$\mathbf{p} = \mu\mathbf{v} - e(\mathbf{r} \times \mathbf{H}_1)\dot{q}/2c, \qquad p_q = L\rho^2\dot{q} + e\mathbf{H}_1\cdot(\mathbf{r} \times \mathbf{v})/2c, \quad (7)$$

denoting by $\mathbf{p}$ the vector $\mathbf{p} = \mathbf{i}p_x + \mathbf{j}p_y + \mathbf{k}p_z$

$$H = \mathbf{p}^2/2\mu + p_q^2/2L\rho^2 - e\mathbf{H}_1\cdot(\mathbf{r} \times \mathbf{p})p_q/2L\rho^2\mu c + U, \qquad (8)$$

neglecting terms of the second order in $\mathbf{H}_1$.

We derive from this expression the wave equation by the Schroedinger variation process[4]

$$\nabla^2\psi + \frac{\mu}{L\rho^2}\frac{\partial^2\psi}{\partial q^2} - e(\mathbf{H}_1 \times \mathbf{r})\cdot\nabla\frac{\partial\psi}{\partial q}/L\ \rho^2 c + \frac{2\mu}{k^2}(E-U)\psi = 0. \quad (9)$$

If we substitute polar coördinates, choosing the direction of the vector $\mathbf{H}_1$ as the polar axis, this becomes

$$[\psi] + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{eH_1}{L\rho^2 c}\frac{\partial^2\psi}{\partial\varphi\partial q} + \frac{\mu}{L\rho^2}\frac{\partial^2\psi}{\partial q^2} = 0. \qquad (10)$$

This is the fundamental equation of our problem and it is entirely different from equation (2). We can easily reduce it to a separable form by using instead of φ a new variable φ' defined by

$$\varphi = \varphi' + q\ eH_1/2\mu c. \qquad (11)$$

The substitution gives

$$[\psi] + \frac{1}{r^2}\frac{1}{\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi'^2} + \frac{\mu}{L\rho^2}\frac{\partial^2\psi}{\partial q^2} = 0. \qquad (12)$$

The term depending on the magnetic field has disappeared, and (12) is formally identical with the equation for the case of a hydrogen atom and

current which do *not* react on each other. Because of this absence of interaction, we can leave the variable q out and treat the phenomena in the space r, ϑ, φ' separately. The problem is then formally identical with that of the unperturbed hydrogen atom in the space r, ϑ, φ. This problem was solved by Schroedinger[4] and all his results are immediately applicable in our case, we need only to substitute everywhere φ' instead of φ. Especially, we must note that the optical frequencies are identical in both cases.

4. Let us now discuss how the phenomena appear in the real physical space. What is the physical meaning of transformation (11)? Since, within the approximation here required, the velocity $\dot{q}$ of the electric charges in our conductor is constant, we have $q = \dot{q}t$, and we had defined $H = H_1\dot{q}$. This gives

$$\varphi = \varphi' + eHt/2\mu c, \tag{13}$$

the familiar expression of the Larmor precession.

We see, therefore, that the discrepancy stated in section 1 is removed. The order of the two operations, introduction of rotating axes of coordinates and Schroedinger procedure, is interchangeable. Both ways we get the same solution of the problem. It hardly is necessary to add that this theory gives the Lorentz triplet and equal intensities for all three components of it.

Equation (2) is incorrect because it is based on the consideration of only a part of our energetic system. It could be obtained from (10) by the assumption of a very special complex dependence of ψ on q, which, without any doubt, is not justified because it does not give standing waves in the $r, \vartheta, \varphi', q$ space.

5. The considerations of the preceding sections suggest a method of extending the Schroedinger-Eckart procedure to the general case of any Hamiltonian system. These authors state their rule for systems subject to the equation (q coördinates of position),

$$H(p, q) - E = 0, \tag{14}$$

in the following form: Substitute in H instead of the momenta p the symbols $ik\partial/\partial q$ and regard H-E as an operator operating on the function ψ. The result is Schroedinger's wave equation in the form

$$[H(ik\partial/\partial q, q) - E]\psi = 0. \tag{15}$$

The rule is enunciated only for Hamiltonian functions not containing the time explicitly, and we have seen that, even in some of these special cases, it fails.

Let us now take the general form of the Hamiltonian equation,

$$H(p, q, t) + \partial S/\partial t = 0. \tag{16}$$

The mathematical beauty of the Hamiltonian theory lies mainly in the fact that the variables of position q and the time t are treated exactly in the same way. From the mathematical point of view the partial $\partial S/\partial t$ is nothing but another momentum pt. It seems, therefore, logical to make for it the same substitution as for the rest of the momenta. The generalized equation we propose is, consequently,

$$[H(ik\partial/\partial q, q, t) + ik\partial/\partial t]\psi' = 0. \tag{17}$$

For instance, in the case of a single electron in a field of conservative forces, this gives

$$\nabla^2\psi' - \frac{2\mu}{k^2} U\psi' + \frac{2\mu}{ik}\frac{\partial\psi'}{\partial t} = 0. \tag{18}$$

We see that Schroedinger's equation, generalized to include the time, does not contain the second derivative with respect to time, but the first. It is, technically speaking, not a wave equation, as he expected it to be, but an equation of conduction with an imaginary conductivity. If we substitute the usual form of the time factor $\psi' = \psi \exp.\ (2\pi i\nu t)$, the result is

$$\nabla^2\psi + \frac{2\mu}{k^2}(h\nu - U)\psi = 0. \tag{19}$$

Comparing with the form (14) we find

$$E = h\nu.$$

This relation, surmised also by Schroedinger, seems to follow in a much more direct and convincing way from our assumption than from Schroedinger's ingenious argumentation.

Our rule stands, also, the test of the Zeeman effect. Equation (2) of section 1 must be replaced now by

$$\nabla^2\psi' + \frac{eH}{ick}\frac{\partial\psi'}{\partial\varphi} + \frac{2\mu}{ik}\frac{\partial\psi'}{\partial t} - \frac{2\mu}{k^2} U\psi' = 0. \tag{20}$$

Substitution (13) transforms this equation into the form (18) with the only difference that it is expressed in the coördinates r, ϑ, φ' instead of r, ϑ, φ. Again we find that the action of a magnetic field is equivalent to a Larmor precession, so that all the conclusions of section 4 hold also for this treatment. It is remarkable that the solution so obtained is rigorous: no terms of the second order are neglected, as in the old theory.

[1] E. Schroedinger, *Ann. Physik*, 79, p. 745 (1926).
[2] C. Eckart, *Physik. Rev.*, October, 1926.
[3] H. A. Lorentz, "Problems of Modern Physics."
[4] E. Schroedinger, *Ann. Physik*, 79, p. 361 (1926).

THE DISPERSION OF ATOMIC HYDROGEN. I—A MEASURE

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The apparatus used in the experiment which is the subject of this communication is related to that named after Puccianti.[1] It consists essentially of identical tubes about 85 cm. long with optically plane parallel ends placed in the arms of an interferometer and filled to the same pressure with hydrogen. A powerful discharge was passed through one of the tubes while "white-light" fringes from a high intensity carbon arc were being observed. The fringes were made horizontal and focussed on the narrow vertical slit of a spectroscope or a very fast spectrograph. The fiducial point was provided in the visual observations by the cross-hairs of the spectroscope and in the photographic by a very fine wire stretched horizontally across the slit of the spectrograph. The field presented was then a bright continuous spectrum traversed by a number of almost horizontal slightly curved and very black fringes and also by the very sharp horizontal shadow of the cross-hairs or the wire across the slit.

The discharge is turned on and fixing attention to a part of the spectrum not disturbed by the Balmer lines which come out in great purity,[2] a shift of the fringes is looked for. Otherwise a series of exposures of 1 to 4 seconds' duration is taken with the discharge alternately off and on and the position of the fringes measured with respect to the fiducial line. This procedure was repeated many times at pressures from 0 to 25 mm. with all varieties of conditions of discharge, purity of hydrogen, etc.

TABLE 1

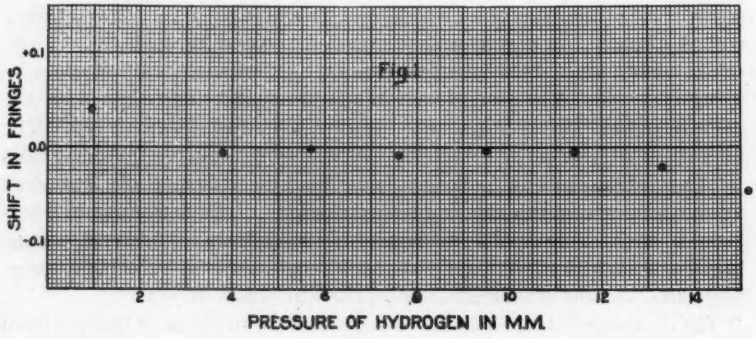
PRESSURE IN MILLIMETERS OF HG MM.	AVERAGE FRINGE SHIFT FRINGES
0.95	+0.040
1.9	−0.015
3.8	−0.006
5.7	−0.001
7.6	−0.009
9.5	−0.004
11.4	−0.005
13.3	−0.022
15.2	−0.047
19	−0.066

The shift of fringes at any wave-length together with a knowledge of the concentration of atomic hydrogen and the index of refraction of molecular hydrogen serves to determine the dispersion of atomic hydrogen.

Of course, there were precautions to take. A complete description of these, together with a discussion of the many troublesome spurious effects

encountered, is being sent to the *Physical Review* for publication. Meanwhile, the experiment is being repeated under conditions having the tremendous advantage of giving practically complete dissociation into atomic hydrogen.

Table 1 gives the average of twelve series of runs from one millimeter pressure to about two centimeters. In each series there were from 5 to 25 determinations of the shift for each pressure. Figure 1 is a graphical representation of table 1.



Each one of these individual determinations as well as every one of countless visual observations is practically the same as the average. The positive shift at the first point, the small negative deflections for intermediate pressures and the negative shift at high pressures are due to well understood disturbing influences; namely—the positive temperature coefficient of the index of refraction of the glass ends—and the negative density gradient from the axis of the tubes to the walls, respectively. Most of the readings were made at a wave-length of about $\lambda = 6000$ Å but a few were taken at different places over the visible spectrum. The conclusion is that with a precision to be considered later the index of refraction of atomic hydrogen is half that of molecular hydrogen or at 1 atmosphere and 0°C. since $\mu - 1$ is small.

$$\mu_H^2 - 1 = \frac{1}{2}(\mu_{H_2}^2 - 1) = 1.36 \times 10^{-4}. \qquad (1)$$

To find the magnitude of the possible error in this figure it is necessary to make an estimate of the partial pressure of atomic hydrogen. Unfortunately, attempts to measure this directly have hitherto been unsuccessful. There are many arguments which may be invoked to decide on a lower limit of the concentration but for the present article we shall restrict ourselves to data obtained from the splendid work of K. F. Bonhoeffer.[3]

In the first place attention is called to the fact that Bonhoeffer's measures of the rate of recombination of atomic hydrogen at 0.5 mm. fit exactly the law

$$\gamma = \gamma_0 e^{-kt} \tag{2}$$

where γ is the fraction of H_2 molecules dissociated and k the so-called specific rate of recombination is put equal to 2. The quantity k must be a function of the pressure which is zero when $p = 0$. Let us say for the time being $k = k_0 p^n$ and then we can write for (2)

$$\gamma = \gamma_0 e^{-2^{n+1} p^n t}, \tag{3}$$

where p is the initial pressure of molecular hydrogen in millimeters.

In our experiments the voltage was supplied from a 16,000 v. 10 kw. transformer with a large condenser in parallel and a variable spark gap in series with the tube. In a few cases the gap was closed giving a 50-cycle sine wave, at other times the gap broke down at least twice a cycle giving condensed discharges certainly less than a thousandth of a second in duration. The interval between discharges was, therefore, never more than one hundredth of a second and usually the time during which the discharge was on, was a negligible part of the cycle. The apparent position of the fringes will correspond to the average value of the dissociation which we proceed to calculate using the most unfavorable assumptions, namely, that the interval between discharges is 0.01 second. We have then

$$\bar{\gamma} = 100\, \gamma_0 \int_0^{\frac{1}{100}} e^{-2^{n+1} p^n t}\, dt \tag{4}$$

$$= \frac{100\, \gamma_0}{2^{n+1} p^n} \left[1 - e^{-\frac{2^{n+1} p^n}{100}}\right]. \tag{4'}$$

The next difficulty is to decide on the pressure dependence of the initial dissociation γ_0. Considering that in the beginning of the region of pressures used, the current through the tube for a given voltage increased with the pressure and considering the facts that at pressures up to about 1 cm., there were often several discharges per cycle and that the power and voltage inputs were increased with the pressure up to about 12 mm., it would be an underestimate at low pressures and an overestimate at high pressures to say that the initial dissociation did not decrease with increase of pressure. Since both of these are unfavorable to the conclusions we are to draw we shall make the assumption with a clear conscience. For the constant of initial dissociation we again refer to Bonhoeffer, who gives, in a very decided underestimate, 20% for the fraction γ_0 at some unmentioned pressure, probably about 0.5 mm. When the 10 kw. transformer and the short duration condensed discharges at 8000 to 16,000 v.

giving current densities of hundreds of amperes per square centimeter—while the discharge passed—(average low frequency currents over the whole cycle went up to several amperes per square centimeter) are compared with the milliamps used by Bonhoeffer, it seems conservative to write

$$\gamma_0 = 0.2$$

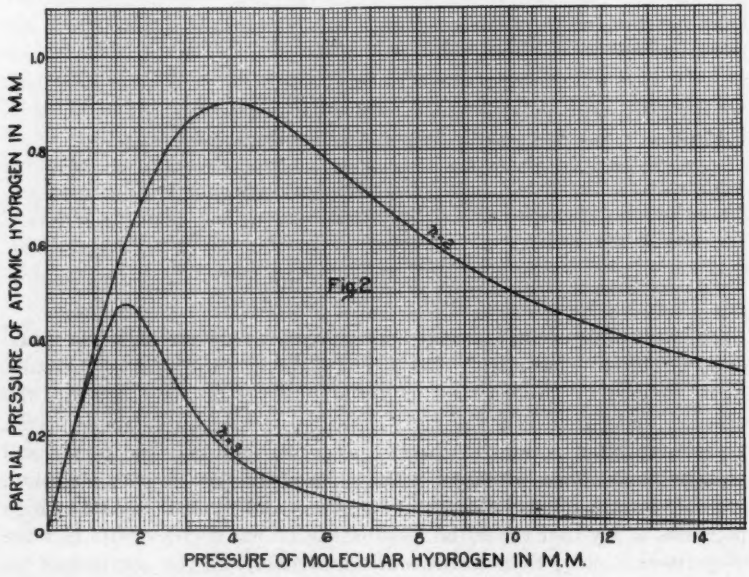
giving from (4′)

$$\bar{\gamma} = \frac{10}{2^n p^n}\left[1 - e^{-\frac{e^{n+1} p^n}{100}}\right]$$

so that for the partial pressure of atomic hydrogen we have, approximately,

$$p_H = \frac{10}{2^{n-1} p_{H_2}^{n-1}}\left[1 - e^{-\frac{2^{n+1} p_{H_2}^n}{100}}\right]. \qquad (5)$$

As to n, the power of the pressure according to which the rate of recombination increases it might be said that for this type of reaction $n = 2$ is plausible, but from the point of view of the chemical kinetics of Born



and Franck,[4] it is well to consider the case $n = 3$. Figure 2 shows the partial pressures of atomic hydrogen as calculated from (5) for each of these values of n.

The maximum partial pressure of H is 0.90 mm. and 0.46 mm. for

$n = 2$ and $n = 3$, respectively. The assumptions made lead, therefore, to no extravagant values of the dissociation.

It remains to estimate how small a shift was detectable. The practised eye notices a sudden shift of 1/30 of a fringe under favorable conditions. The measurement of a photographic plate is still better, yet we shall take it that the average values given in table 1 are reliable to 0.03 fringe. In that case, considering the disturbing influences already mentioned the fringe shift is zero for all pressures and even without allowing for these influences it is zero for the pressures where the partial pressure of atomic hydrogen is appreciable. The expression for the fringe shift ΔN caused by the formation of atomic hydrogen when the discharge is turned on may be written

$$\Delta N = K[(\mu_{H_2}^2 - 1)\Delta p_{H_2} + (\mu_H^2 - 1)\Delta p_H] \qquad (6)$$

where μ_{H_2} and μ_H are the refractive indices of H_2 and H, respectively, K is a constant which calculated for case at hand is $\frac{1}{1.9(\mu_{H_2} - 1)}$ and where Δp_{H_2} and Δp_H are the changes in pressure of the two components due to the discharge. These satisfy the relation

$$\Delta p_H = -2\Delta p_{H_2}.$$

Putting $\Delta N = 0.03$ there results from (6)

$$(\mu_H^2 - 1) = [1.36 \pm 0.17] \times 10^{-4} \qquad (7)$$

if $n = 2$, and

$$(\mu_H^2 - 1) = [1.36 \pm 0.34] \times 10^{-4} \qquad (8)$$

if $n = 3$.

This is very crude measurement. Nevertheless, it is sufficient to discriminate against some of the dispersion formulae that have been proposed. This matter is considered in the next paper where it is shown that the formulae which are satisfactory in that they put the anomalous dispersion at the observed frequencies yield values for $(\mu_H^2 - 1)$ which differ from that given by experiment, by several times the margin indicated in (8).

It is realized that the matter cannot be regarded as settled until it is possible to take direct observations of the amount of atomic hydrogen while the dispersion is being measured. The author hopes to attempt this at the Naval Research Laboratory in Washington, D. C.

[1] L. Puccianti, *Il. Nuovo Cimento*, **2**, 257 (1901). Similar arrangements have been used by D. Roschdestwensky, *Ann. Physik.*, **39**, 307 (1912); R. Ladenburg and St. Loria, *Verh. D. Physik. Ges.*, **10**, 858 (1908) and quite recently by R. Ladenburg, H. Kopfermann and A. Carst, *Sitz. Preuss. Akad. Wissenschaften*, **20–22**, 255 (1926).

[2] It might be mentioned for what it is worth that under favorable conditions, with the Balmer series horribly overexposed for a width of from 50 to 200 Å. not a trace of the secondary spectrum could be seen. A few Hg and Al lines beyond the violet and a

very noticeable continuous spectrum were present. Using a moderately narrow slit, the secondary spectrum could be found visually only by those who knew just where to look for it.

[3] K. F. Bonhoeffer, *Zeit. Physik. Chem.*, **113**, 199 (1924).

[4] M. Born and J. Franck, *Ann. Physik*, **76**, 225 (1925) and *Zeit. Physik*, **31**, 411 (1925).

THE DISPERSION OF ATOMIC HYDROGEN II—A CALCULATION

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In the preceding paper experimental evidence is brought forward to show that at a wave-length $\lambda = 6000$ Å the index of refraction of atomic hydrogen is $\mu_H = 1.000068$. Expressions for this quantity derived by Sommerfeld and Epstein* check this value but cannot be taken as correct because they all require anomalous dispersion at frequencies corresponding to the mechanical frequencies of the model and nothing in particular at the spectral frequencies. It is significant that these classical computations using atomic models given by the quantum theory should give a correct result for long wave-lengths. On the other hand—as shown in this paper—attempts to adjust the dispersion formulae to give the proper frequencies for anomalous dispersion lead to a numerical value for μ_H which is decidedly too large.

The author has calculated and will publish elsewhere the perturbations of a known system by a force of the form $E_0F(t)$. As a special case we have the problem of dispersion—namely, an atomic system acted on by a plane wave. In this case we have an electric field $E_0 \cos 2\pi\nu t$. If general cylindrical coördinates are chosen with the z direction parallel to the electric field, the expression for an element of the first order perturbation of the $2f$ dimensional matrix $\mathbf{q}_z$ is given by[1]

$$q_z^1(n_1 \ldots n_f, m_1 \ldots m_f) = \frac{eE_0 \cos 2\pi\nu t}{2\pi i h m} \sum_{k_1 \ldots k_f} \frac{q_z^0(n_1 \ldots n_f, k_1 \ldots k_f) p_z^0(k_1 \ldots k_f, m_1 \ldots m_f) - p_z^0(n_1 \ldots n_f, k_1 \ldots k_f) q_z^0(k_1 \ldots k_f, m_1 \ldots m_f)}{(\nu_0(n_1 \ldots n_f, k_1 \ldots k_f) + \nu)(\nu_0(k_1 \ldots k_f, m_1 \ldots m_f) + \nu)} \quad (1)$$

where f is the number of degrees of freedom of the system, m the mass of the electron and $\mathbf{q}_z^0$, $\mathbf{p}_z^0$ are the unperturbed canonical matrices corre-

sponding to the cylinder coördinate z. The summation extends over all combinations of the k's excepting that which makes $\nu_0(n_1 \ldots n_f, k_1 \ldots k_f) = 0$. The quantities $\nu_0(n_1 \ldots n_f, k_1 \ldots k_f)$, $\nu_0(k_1 \ldots k_f, n_1 \ldots n_f)$ are the frequencies characteristic of changes to or from the state $(n_1 \ldots n_f)$. If n_i is the main quantum number then for $n_i > k_i$ we have an emission and for $n_i < k_i$ an absorption. The elements of the matrix $\mathbf{q}_z^1$ which are interesting for dispersion are those with the same frequency as that of the incident light so that $\nu_0(n_1 \ldots n_f, m_1 \ldots m_f) = 0$, viz., the diagonal terms

$$q_z^1(n_1 \ldots n_f,\ n_1 \ldots n_f) = -\frac{eE_0 \cos 2\pi\nu t}{2\pi i h m} \sum_{k_1 \ldots k_f} \frac{q_z^0(n_1 \ldots n_f, k_1 \ldots k_f) p_z^0(k_1 \ldots k_f,\ n_1 \ldots n_f) - p_z^0(n_1 \ldots n_f,\ k_1 \ldots k_f) q_z^0(k_1 \ldots k_f,\ n_1 \ldots n_f)}{\nu_0(n_1 \ldots n_f,\ k_1 \ldots k_f)^2 - \nu^2}. \tag{2}$$

If we separate the emission from the absorption terms and note that the product

$$p_z^0(n_1 \ldots n_f,\ k_1 \ldots k_f) q_z^0(k_1 \ldots k_f,\ n_1 \ldots n_f)$$

has numerically opposite signs in the two cases we have

$$q_z^1(n_1..n_f,\ n_1..n_f) = -\frac{eE_0 \cos 2\pi\nu t}{2\pi i h m} \left\{ \sum_{n_i < k_i} \frac{p_z^0(n_1..n_f, k_1..k_f) q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f) p_z^0(k_1..k_f, n_1..n_f)}{\nu_0(n_1..n_f, k_1..k_f)^2 - \nu^2} - \sum_{n_i > k_i} \frac{p_z^0(n_1..n_f, k_1..k_f) q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f) p_z^0(k_1..k_f, n_1..n_f)}{\nu_0(n_1..n_f, k_1..k_f)^2 - \nu^2} \right\}, \tag{3}$$

where only absolute magnitudes of the two summations are to be taken into account.

To arrive at an expression for the index of refraction of all the atoms in the state $(n_1 n_2 \ldots n_f)$ we use the relation

$$\mu^2 - 1 = \frac{4\pi P N}{E_0 \cos 2\pi\nu t} = \frac{4\pi N e}{E_0 \cos 2\pi\nu t} q_z^1(n_1 \ldots n_f,\ n_1 \ldots n_f).$$

Where P is the polarization due to the electric field and N is the number of atoms per cc. in the state $(n_1 \ldots n_f)$. Remembering that z being a cylinder coördinate $\mathbf{p}_z = m\dot{\mathbf{q}}$ and referring to the work of Kramers and Heisenberg[2] and also that of Born and Jordan[3] it is seen that the matrix mechanics expression as derived from (3) namely

$$(\mu^2 - 1)_M = -\frac{2e^2N}{ihm}$$

$$\left\{ \sum_{n_i<k_i} \frac{p_z^0(n_1..n_f, k_1..k_f)q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f)p_z^0(k_1..k_f, n_1..n_f)}{\nu_0(n_1..n_f, k_1..k_f)^2 - \nu^2} \right.$$
$$\left. - \sum_{n_i>k_i} \frac{p_z^0(n_1...n_f, k_1..k_f)q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f)_z^0(k_1..k_f, n_1..n_f)}{\nu_0(n_1..n_f, k_1..k_f)^2 - \nu^2} \right\} \quad (4)$$

has the same satisfactory form as that given by Kramers[4]

$$(\mu^2 - 1)_K = \frac{e^2N}{\pi m} \left\{ \sum_a \frac{f_a}{\nu_a^2 - \nu^2} - \sum_e \frac{f_e}{\nu_e^2 - \nu^2} \right\}. \quad (5)$$

To those who had been following the work of Kuhn, Thomas, Reiche and Heisenberg[5] leading to the matrix mechanics it was not surprising that the condition

$$\mathbf{p}\,\mathbf{q} - \mathbf{q}\,\mathbf{p} = \frac{h}{2\pi i}\,\mathbf{1}$$

should give a result quite analogous to that of Kramers, but it is not obvious that the two theories should give identical numerical values for the index of refraction.

Using the results in a very interesting paper of Reiche and Thomas[6] we can write for the case of atomic hydrogen in the normal state

$$\sum_a f_a - \sum_e f_e = 1. \quad (6)$$

For the corresponding part of (4) we have

$$\sum_{n_i<k_i} p_z^0(n_1..n_f, k_1..k_f)q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f)p_z^0(k_1..k_f, n_1..n_f)$$
$$- \sum_{n_i>k_i} p_z^0(n_1..n_f, k_1..k_f)q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f)p_z^0(k_1..k_f, n_1..n_f)$$
$$= \sum_{k_1...k_f} p_z^0(n_1..n_f, k_1..k_f)q_z^0(k_1..k_f, n_1..n_f) - q_z^0(n_1..n_f, k_1..k_f)p_z^0(k_1..k_f, n_1..n_f) = \frac{h}{2\pi i}, \quad (6')$$

showing that in this case at least the numerical values of μ calculated from (4) and (5) are equal.

With the possibility of an experimental verification at hand it becomes interesting to make this calculation. And here we appreciate the strength of Kramers' dispersion formula—for not only is it difficult to check experimentally but the numerical computation is none too simple. For the sums (6) and (6′) and not the same as those appearing in (4) and (5).

In fact until reliable "transition-probability coefficients" are obtained the theory is almost safe. Not quite, however, for we can find a lower bound for μ_H by noticing that since for the normal state the series is the Lyman series

$$\nu_a = R\left(1 - \frac{1}{a^2}\right) < R,$$

so that

$$\frac{1}{\nu_a^2 - \nu^2} > \frac{1}{\nu_a^2} > \frac{1}{R^2},$$

and

$$\sum_a f_a = 0$$

it follows from (4) or (5) using (6) or (6′)

$$(\mu_H^2 - 1) = \frac{e^2N}{\pi m}\sum_a \frac{f_a}{\nu_a^2 - \nu^2} > \frac{e^2N}{\pi m R^2}\sum_a f_a. \qquad (7)$$

This gives for both theories for atomic hydrogen at 1 atmosphere and 0°C.

$$\mu_H^2 - 1 > 2.03 \times 10^{-4}, \qquad (8)$$

whereas experiments by the author done at the California Institute of Technology and discussed in the preceding paper of these PROCEEDINGS show (taking the extreme estimate of the error),

$$(\mu_H^2 - 1)_{exp.} \doteq [1.36 \pm 0.34] \times 10^{-4}. \qquad (9)$$

The discrepancy is really much more serious. For Schrödinger[7] has recently calculated the values of f_a for the Balmer series on the basis of his formulation of the quantum theory and obtained in agreement with the empirical results due to Ladenburg[8] the relative values

$$f_{H_\alpha} : f_{H_\beta} : f_{H_\gamma} : f_{H_\delta} = 1.281 : 0.2386 : 0.08975 : 0.04418.$$

If it is assumed that the same proportions hold for the Lyman series we get instead of (7) for visible light where we can take $\nu \doteq 0$

$$(\mu_H^2 - 1) \doteq \frac{e^2N}{\pi m R^2} \cdot \frac{1}{165}\left[\frac{128}{(3/4)^2} + \frac{24}{(8/9)^2} + \frac{9}{(15/16)^2} + \frac{4}{(24/25)^2}\right], \qquad (10)$$

giving to compare with (9)

$$(\mu_H^2 - 1) \doteq 3.38 \times 10^{-4}. \qquad (11)$$

On the other hand, if we allow that the compatibility between theory and experiment in the case of the Balmer series establishes Pauli's direct calculation on the Lyman series mentioned at the end of Schrödingers' paper we have instead of (10)

$$(\mu_H^2 - 1) \doteq \frac{e^2N}{\pi mR^2}\,\frac{1}{801}\left[\frac{625}{(3/4)^2} + \frac{119}{(8/9)^2} + \frac{42}{(15/16)^2} + \frac{21}{(24/25)^2}\right] \quad (10')$$

giving to compare with (9)

$$(\mu^2 - 1) = 3.34 \times 10^{-4}. \quad (11')$$

If the improved experiments now in progress verify the violent disagreement between (8), (11) or (11′) and (9) it will not be sufficient to search for a modification of the perturbation theory of the matrix calculus. The difficulty apparently strikes through the Kramers' dispersion theory at the very foundation of the matrix mechanics, viz., the Correspondence Principle. Or else, since in this simple case the original work of Ladenburg[9] leads to the very expression derived by Kramers in his development of that work, we begin to suspect that there may be something wrong with our notion as to the nature of the phenomena of dispersion.

[*] A. Sommerfeld—Elster and Geitel—"Festschrift," p. 575 (1916), and P. S. Epstein, *Zeit. Physik*, **9**, 93 (1922).

[1] The equivalent expression for the case of systems of one degree of freedom and using cartesian coördinates was given by M. Born, W. Heisenberg and P. Jordan, *Zeit. Physik*, **35**, 572 (1926), and by Born in his *Problems of Atomic Dynamics*, page 90. The factor 2 by which the latter differs is apparently a typographical error.

[2] H. A. Kramers and Heisenberg, *Zeit. Physik*, **31**, 693 (1925).

[3] M. Born and P. Jordan, *Zeit. Physik*, **34**, 887 (1925).

[4] H. A. Kramers, loc. cit., also *Nature*, **113**, 673 (1924) and **114**, 310 (1924).

[5] W. Kuhn, *Zeit. Physik*, **33**, 408 (1925). W. Thomas, *Naturwissenschaften*, **13**, 627 (1925). F. Reiche and W. Thomas, *Zeit. Physik*, **34**, 510 (1925). W. Heisenberg, *Zeit. Physik*, **33**, 879 (1925).

[6] F. Reiche and W. Thomas, loc. cit.

[7] E. Schrödinger, *Ann. Physik* (4), **80**, 477 (1926).

[8] R. Ladenburg, *Ann. Physik* (4), **38**, 249 (1912).

[9] R. Ladenburg, *Zeit. Physik*, **4**, 451 (1921); also R. Ladenburg and F. Reiche, *Naturwissenschaften*, **1923**, p. 584.

THE DEVELOPMENT OF MENDELIAN CHARACTERS IN APLOCHEILUS LATIPES*

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The Japanese freshwater fish, *Aplocheilus latipes*, offers certain special advantages for the study of the development of Mendelian characters. A genetic analysis made by Aida[1] has shown a clearly defined Mendelian inheritance of color characteristics. Aplocheilus is oviparous while other fish of which such an analysis has been made are viviparous. Hence this is the only fish of which the genetics is in part known and which readily permits of a study of the development of Mendelian characters from fertilization to the adult condition. Also the color effects are produced by pigment cells which may be identified early in development.

The genetic constitution of the color types here considered may be indicated by the following formulae based on the analysis made by Aida:

Wild (homozygous)	BBYY
Yellow (homozygous)	bbYY
White (homozygous)	bbyy
Variegated (homozygous)	B'B'YY

Microscopic studies of the scales of these fish have shown that the colors are produced by three types of pigment cells. These are the melanophores, the brown chromatophores and the yellow chromatophores or xanthophores. The cells may also be differentiated by their characteristic form and by the relative solubility of the contained pigment. In ninety-five per cent alcohol the melanin is practically insoluble while the pigment of the brown chromatophores will dissolve in about eight minutes and that of the yellow chromatophores in three hours.

The Mendelian effects are due to the varied development of the melanophores and the xanthophores while the brown chromatophores remain nearly constant in all types. A study of the pigment cells of the wild type (BBYY) shows that the color, a greenish brown, is produced by the presence of both melanophores and xanthophores. Superficial inspection of the yellow type (bbYY) seems to indicate that only xanthophores are present. It is, however, possible by special treatment to show that melanophores with slight development of pigment are abundantly present. The method used was to remove scales from fish anaesthetized by chloretone and to treat these while fresh with a strong solution of adrenalin. This causes a concentration of pigment in cells rendering faintly colored cells more visible. These scales were then fixed in 95% alcohol which dissolved pigment in

cells other than the melanophores thus rendering confusion impossible. The scales were mounted in balsam and counts of the melanophores within a uniform measured area from each scale were made. The results showed that the melanophores were present in the scales of the yellow fish in numbers comparable with those of the wild type. Similar studies of the white fish (bbyy) demonstrated the presence of melanophores in abundance. These observations are summarized in the chart, figure 1, in which are plotted the results of counts of chromatophores from 100 scales of each type of fish obtained from the region of the pectoral fin. Here it will be noted that the modal class of chromatophore counts is the same for all three types but that more scales with larger counts are found among the yellow and white types than in the wild type. Apparently the large size of the in-

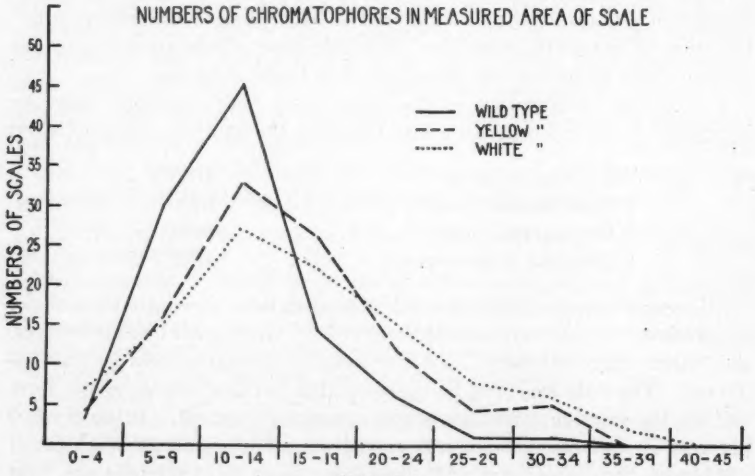

FIGURE 1

dividual cells in the wild form limits the number which occupy a given area. Also an examination under oil-immersion lens of fresh scales from white fish revealed the presence of a network of cells similar in position, form and size of granules, to the xanthophores of the yellow and wild types but usually without any yellow pigment. It is thus indicated that the Mendelian gene produces its effect, not by controlling the number of pigment cells produced but by regulating the amount of pigment in cells always present.

Studies on earlier stages in the life history have been made. Young fish were examined shortly after hatching and counts of melanophores visible on the dorsal surface were made. The earlier counts seemed to indicate that the color effects were correlated with different numbers of

chromatophores rather than with the development of pigment in cells. The wild type showed the greatest number of melanophores, the variegated less and the yellow the least. Here again, however, treatment with adrenalin revealed that there were approximately equal numbers of melanophores in all three types. The contraction of pigment in cells otherwise invisible showed their presence in areas which previously seemed unoccupied.

Observations on embryonic stages have shown that in the case of crosses between wild types heterozygous for the black factor (BbYY) and yellows (bbYY) that the Mendelian ratios could be distinguished at two days after fertilization or at the stage of about eleven somites. Examples of this are the counts of eggs obtained from an aquarium containing one wild (heterozygous) male and two white females. Of 206 eggs 104 showed pigmentation and 102 were unpigmented. Table 1 shows counts of melanophores in eggs of a single spawning from a similar mating.

TABLE 1

COUNT OF CHROMATOPHORES FROM SINGLE BATCH OF EGGS THREE DAYS AFTER FERTILIZATION FROM CROSS OF YELLOW FEMALE BY WILD MALE (HETEROZYGOUS)

Body region	22	20	16	20	25	24	12	19	20	0	0	0	0	0	0	0
Tail region	25	22	14	27	19	22	24	20	38	0	0	0	0	0	0	0
Yolk (dorsal)	34	20	33	40	27	37	30	32	35	0	0	0	0	0	0	0
Yolk (ventral)	34	24	36	27	28	60	32	32	20	1	0	0	1	0	2	0
Total	115	86	99	114	99	143	98	103	113	1	0	0	1	0	2	0

As far as observations have been pushed back in embryonic history the three types of pigment cells remain distinct. There has been no indication that one type may arise from another. This indicates that the differentiation must take place at a stage in cell lineage proceeding the formation of pigment.

Studies of scales of variegated fish show results in harmony with those of the preceding types discussed. Faintly pigmented melanophores are present in the light areas in numbers equal to those of the more deeply pigmented types in dark areas. This condition seems to indicate a physiological differentiation between the darker and the lighter melanophores, that is, that some types possess a greater melanin-producing power than others. This lends weight to the suggestion that a similar distinction exists between the melanophores of the wild and the yellow-white types. A hormone activity might account for the latter case but could hardly be construed to be a complete explanation of the development of a variegated pattern as hormones would be carried to all parts alike.

All of the observations seem in harmony with the hypothesis that the Mendelian gene produces its effect not by controlling "division energy," regulating numbers of cells but by controlling the amount of melanin

to be elaborated in cells always present and further, that the definitive action is at some stage in cell lineage preceding the earliest appearance of pigment.

* This investigation has been aided by a grant from the BACHE FUND OF THE NATIONAL ACADEMY OF SCIENCES.

[1] Aida, T., *Genetics*, Baltimore, 6, 554–73 (1921).